INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 12, 825–833 (1991)

A CHEBYSHEV SPECTRAL COLLOCATION METHOD USING A STAGGERED GRID FOR THE STABILITY OF CYLINDRICAL FLOWS

MEHDI R. KHORRAMI

Department of Mechanical Engineering and Mechanics, Old Dominion University, Norfolk, VA 23529-0247, U.S.A.

SUMMARY

A staggered spectral collocation method for the stability of cylindrical flows is developed. In this method the pressure is evaluated at different nodal points than the three velocity components. These modified nodal points do not include the two boundary nodes; therefore the need for the two artificial pressure boundary conditions employed by Khorrami *et al.* is eliminated. It is shown that the method produces very accurate results and has a better convergence rate than the spectral tau formulation. However, through extensive convergence tests it was found that elimination of the artificial pressure boundary conditions does not result in any significant change in the convergence behaviour of spectral collocation methods.

KEY WORDS Spectral collocation Chébyshev polynomials Staggered grid Pressure boundary conditions Hydrodynamic stability Cylindrical flows

1. INTRODUCTION

Velocity and pressure can be evaluated at staggered grid locations to simplify pressure boundary condition requirements for a variety of incompressible fluid dynamic applications. Unstaggered methods can be programmed more easily, but they often require pressure boundary conditions which can only be generated by manipulation of the equations of motion, rather than by direct application of a physical requirement. Recently, Khorrami *et al.*[1] (hereinafter referred to as KMA) have used unstaggered spectral collocation (pseudospectral) methods in studying the stability of a range of viscous, swirling flows. They showed that suitable pressure boundary conditions could be developed by manipulating the conservation-of-momentum equations and that accurate eigenvalues could be obtained in an efficient manner. However, they did not investigate staggered methods, and since the indirect pressure boundary conditions are at least unattractive, if not controversial, questions about the utility of staggered and unstaggered formulations remain. The purpose of this work is to present a consistent formulation of the governing equations required to implement staggered spectral collocation studies of the stability of viscous, swirling flows and examine the staggered implementation with respect to the previous unstaggered study of KMA.

In the unstaggered method of KMA the flow variables were expanded in terms of a truncated Chebyshev series. The global eigenvalues of the discretized system were then obtained by a generalized complex QZ routine.[2] They showed that the resulting algorithm was robust and easy to implement while being efficient. The advantages of the spectral collocation method over similar (spectral tau and spectral Galerkin) schemes were twofold. First, it was easily extendable to compressible flows, since the coefficients of the flow variables are always evaluated in the physical

0271–2091/91/090825–09$05.00

Received 26 July 1989
Revised 11 June 1990

space. Secondly, the use of the collocation formulation greatly simplified the implementation of boundary conditions and any new basic flow or co-ordinate transformation.

The set of governing equations obtained by KMA are of sixth order. Therefore the six boundary conditions derived for the three components of the velocity are sufficient for a well-posed boundary value problem. However, a straightforward discretization, as employed in their work, resulted in a system which required two 'artificial' pressure boundary conditions, even though a Poisson equation is not involved in the calculations. In KMA, Neumann conditions were prescribed on the pressure at the two boundaries. This pressure gradient condition is well known whenever the Navier–Stokes equations are solved in primitive variables and has been discussed by Orszag and Israeli[3] and Gresho and Sani.[4] The prescribed conditions are obtained by taking the inner product of the vector momentum equation with the unit normal for the boundary. In cylindrical–polar co-ordinates this turns out to be the radial momentum equation in its original form, evaluated at the boundaries. KMA showed that this formulation was viable and produced eigenvalue accuracy which was comparable to other methods.

Some controversy surrounds the indirect pressure boundary conditions, particularly as they apply to Poisson solvers. Strikwerda[5] questioned the validity of these boundary conditions for finite difference methods. Most of the questions related to finite difference methods have been answered by Gresho and Sani[4] and Roache.[6] While a staggered Chebyshev spectral collocation method has been employed by others (e.g. Malik *et al.*[7] and Montigny-Rannou and Morchoisne[8]) for attacking Navier–Stokes equations and by Macaraeg *et al.*[9] for solving the linear stability of high-speed shear layers, the boundary condition implications for hydrodynamic stability analyses and a direct comparison between staggered/unstaggered formulations have received little attention.

Since numerical hydrodynamic stability calculations are very sensitive to both the types of boundary conditions and how they are implemented, the behaviour of staggered and non-staggered spectral methods is an important concern. Owing to the global nature of the spectral methods, convergence rates are very sensitive to boundary conditions. Hence it is important to remove any ambiguity associated with the indirect pressure boundary conditions and to determine whether a staggered approach improves convergence.

In this study we have staggered the pressure grids and therefore eliminated the need for the two indirect pressure boundary conditions. The formulation is developed in detail in the upcoming sections. As mentioned by KMA, this results in a somewhat more involved procedure. The convergence rate and accuracy of the present analysis will then be compared with other spectral methods and the KMA study. For simplicity, the notation employed in KMA will be used here and the reader is referred to that paper for additional information.

2. STABILITY PROBLEM

Assuming three-dimensional perturbations of the type

$$\{u, v, w, p\} = \{\mathrm{i}F(r), G(r), H(r), P(r)\}\ \mathrm{e}^{\mathrm{i}(\alpha z + n\theta - \omega t)}, \tag{1}$$

the linearized form of the governing equations and the boundary conditions in cylindrical–polar co-ordinates (r, θ, z) are given by KMA and will not be presented here. However, for the present study there was a slight modification to the form of their boundary conditions applied at the centreline as explained below.

In the case when the azimuthal wave number $|n| = 1$, two of the boundary conditions become linearly dependent. In KMA another relation was deduced by enforcing the continuity equation on the centreline, resulting in

$$2F'(0)+nG'(0)=0. \tag{2}$$

However, from a careful examination of the radial and azimuthal momentum equations as $r \to 0$, it can be shown that this condition (equation (2)) is identical to the conditions

$$F'(0)=0 \tag{3}$$

or

$$G'(0)=0. \tag{4}$$

Therefore in the present study equation (3) is used as the appropriate boundary condition for $|n|=1$.

3. CHEBYSHEV SPECTRAL COLLOCATION FOR STAGGERED GRID

The unique properties of Chebyshev polynomials in developing hydrodynamic stability codes have been elaborated by Orszag[10] and Gottlieb and Orszag.[11] Also, function expansions for Chebyshev spectral collocation and explicit set-up of the derivative matrices for Gauss–Lobatto points are given in detail by Gottlieb *et al.*[12] Therefore in this study attention is focused on the development of the interpolant polynomial and derivative matrices for the staggered nodes. Here it suffices to mention that the collocation points based on Gauss–Lobatto quadrature, ξ_j (where the velocity components and the three momentum equations are evaluated), are the extrema of the last retained Chebyshev polynomial ($T_N(\xi)$) in the truncated series. These points are defined by

$$\xi_j=\cos\left(\frac{\pi j}{N}\right), \quad j=0, 1, \ldots, N, \tag{5}$$

where the centreline and outer wall boundaries correspond to $j=0$ and N respectively. Furthermore, if $\bar{A}_{jk}$ are the elements of the first-derivative matrix evaluated at the ξ_j-nodes, then the second-derivative matrix can be constructed via

$$\bar{B}_{jk}=\bar{A}_{jm}\,\bar{A}_{mk}. \tag{6}$$

The pressure and the continuity equation are evaluated at the collocation points $\xi_{j+1/2}$ which are the roots of $T_N(\xi)$ and are given by

$$\xi_{j+1/2}=\cos\left(\frac{(2j+1)\pi}{2N}\right), \quad j=0, 1, \ldots, N-1. \tag{7}$$

Note that this set of points does *not* include the two boundary points. Next, the pressure is represented with an interpolating polynomial of degree $N-1$ (rather than N). Therefore we write

$$P(\xi)=\sum_{j=0}^{N-1} h_j(\xi)\,P(\xi_{j+1/2}), \tag{8}$$

where the interpolant $h_j(\xi)$ is given as

$$h_j(\xi)=\frac{(-1)^j\sin(\phi_{j+1/2})\,T_N(\xi)}{N(\xi-\xi_{j+1/2})}, \quad j=0, 1, \ldots, N-1. \tag{9}$$

It can easily be shown that

$$h_j(\xi_{k+1/2})=\delta_{jk}, \tag{10}$$

where δ_{jk} is the usual Kronecker delta.

At this time, two sets of interpolating matrices are needed to interpolate from the staggered points to the grid points and vice versa. That is,

$$F_{j+1/2}=\sum_{k=0}^{N} M^{*}_{jk} F_k, \qquad j=0, 1, \ldots, N-1, \tag{11}$$

and

$$P_j=\sum_{k=0}^{N-1} M_{jk} P_{k+1/2}, \qquad j=0, 1, \ldots, N, \tag{12}$$

where the elements of the matrix $\mathbf{M}^*$ are given as

$$M^{*}_{jk}=\frac{(-1)^{j+k+1}(1-\xi^2_{j+1/2})^{1/2}}{C_k N(\xi_{j+1/2}-\xi_k)}, \quad j=0, 1, \ldots, N-1, \quad k=0, 1, \ldots, N, \tag{13}$$

and that of $\mathbf{M}$ as

$$M_{jk}=\frac{(-1)^{j+k}(1-\xi^2_{k+1/2})^{1/2}}{N(\xi_j-\xi_{k+1/2})}, \qquad j=0, 1, \ldots, N, \quad k=0, 1, \ldots, N-1. \tag{14}$$

In order for $\mathbf{M}^*$ to be a square matrix, an extra row ($j=N$) with zero elements is added. Similarly, in the case of $\mathbf{M}$ a column ($k=N$) is added which contains null elements.

In the case of the derivatives there are two different ways to interpolate. In one method the explicit functions g_j (which represent the interpolating polynomial for the velocity components) and h_j are first differentiated and then evaluated at the required points. Hence the derivative matrices can be constructed explicitly. An equally valid method is to employ equation (11). That is, if

$$\left.\frac{\mathrm{d}F}{\mathrm{d}\xi}\right|_{j+1/2}=\sum_{k=0}^{N} \bar{A}^{*}_{jk} F_k, \quad j=0, 1, \ldots, N-1, \tag{15}$$

then

$$\bar{A}^{*}_{jk}=M^{*}_{jm}\bar{A}_{mk}. \tag{16}$$

Here we have used a mix of the two methods. That is, if

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\xi}\right|_{j}=\sum_{k=0}^{N-1} \frac{\mathrm{d}h_k(\xi_j)}{\mathrm{d}\xi} P_{k+1/2}, \quad j=0, 1, \ldots, N, \tag{17}$$

then we employ

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\xi}\right|_{j}=\sum_{k=0}^{N-1} E_{jk} P_{k+1/2}, \tag{18}$$

where the elements of $\mathbf{E}$ are given as

$$E_{jk}=\frac{(-1)^{j+k+1}(1-\xi^2_{k+1/2})^{1/2}}{N(\xi_j-\xi_{k+1/2})^2}, \qquad j=1, 2, \ldots, N-1, \qquad k=0, 1, \ldots, N-1, \tag{19}$$

$$E_{0k}=\frac{(-1)^{k}(1-\xi^2_{k+1/2})^{1/2}}{N}\left(\frac{N^2}{1-\xi_{k+1/2}}-\frac{1}{(1-\xi_{k+1/2})^2}\right), \qquad k=0, 1, \ldots, N-1,$$

$$E_{Nk}=\frac{(-1)^{k+N}(1-\xi^2_{k+1/2})^{1/2}}{N}\left(\frac{N^2}{1+\xi_{k+1/2}}-\frac{1}{(1+\xi_{k+1/2})^2}\right), \qquad k=0, 1, \ldots, N-1.$$

A null column ($k=N$) is added to make $\mathbf{E}$ a square matrix.

Finally, if the scaling factor for transformation between the physical and computational domain is given as

$$S_j = \left.\frac{\mathrm{d}\xi}{\mathrm{d}r}\right|_j, \quad j=0, 1, \ldots, N,$$

then the first-derivative matrix $\bar{\mathbf{A}}$ in the physical domain may be written as

$$A_{jk} = S_j \bar{A}_{jk} \tag{20}$$

$$A^*_{jk} = M^*_{jm} A_{mk}, \tag{21}$$

with similar relationships holding in the case of the mid-cell points.

Employing all of the relations developed above, the governing equations, in discretized form, are:

continuity,

$$\sum_{k=0}^{N} A^*_{jk} F_k + \frac{2}{1-\xi_{j+1/2}} \sum_{k=0}^{N} M^*_{jk} F_k + \frac{2n}{1-\xi_{j+1/2}} \sum_{k=0}^{N} M^*_{jk} G_k + \alpha \sum_{k=0}^{N} M^*_{jk} H_k = 0; \tag{22}$$

r-momentum,

$$\sum_{k=0}^{N} B_{jk} F_k + \left(\frac{2}{1-\xi_j} - Re\, U_j\right) \sum_{k=0}^{N} A_{jk} F_k + \left(\mathrm{i}\, Re\,\omega - Re\, S_j \left.\frac{\mathrm{d}U}{\mathrm{d}\xi}\right|_j - \frac{\mathrm{i}2 Re\, n\, V_j}{1-\xi_j} - \frac{4(n^2+1)}{(1-\xi_j)^2}\right) F_j$$
$$-\left(\frac{4(2n)}{(1-\xi_j)^2} + \frac{2(\mathrm{i}\,2 Re\, V_j)}{1-\xi_j}\right) G_j + \mathrm{i}\, Re \sum_{k=0}^{N} E_{jk} P_{k+1/2} - \mathrm{i}\alpha\, Re\, W_j F_j - \alpha^2 F_j = 0; \tag{23}$$

θ-momentum,

$$-\left(\mathrm{i}\, Re\, S_j \left.\frac{\mathrm{d}V}{\mathrm{d}\xi}\right|_j + \frac{4(2n)}{(1-\xi_j)^2} + \frac{2(\mathrm{i} Re\, V_j)}{1-\xi_j}\right) F_j + \sum_{k=0}^{N} B_{jk} G_k + \left(\frac{2}{1-\xi_j} - Re\, U_j\right) \sum_{k=0}^{N} A_{jk} G_k$$
$$+\left(\mathrm{i}\, Re\,\omega - \frac{2(\mathrm{i} Re\, n\, V_j)}{1-\xi_j} - \frac{2(Re\, U_j)}{1-\xi_j} - \frac{4(n^2+1)}{(1-\xi_j)^2}\right) G_j$$
$$-\frac{2(\mathrm{i} Re\, n)}{1-\xi_j} \sum_{k=0}^{N} M_{jk} P_{k+1/2} - \mathrm{i}\alpha\, Re\, W_j G_j - \alpha^2 G_j = 0; \tag{24}$$

z-momentum,

$$-\mathrm{i}\, Re\, S_j \left.\frac{\mathrm{d}W}{\mathrm{d}\xi}\right|_j F_j + \sum_{k=0}^{N} B_{jk} H_k + \left(\frac{2}{1-\xi_j} - Re\, U_j\right) \sum_{k=0}^{N} A_{jk} H_k$$
$$+\left(\mathrm{i}\, Re\,\omega - \frac{2(\mathrm{i} Re\, n\, V_j)}{1-\xi_j} - \frac{4n^2}{(1-\xi_j)^2}\right) H_j - \mathrm{i}\alpha\, Re\, W_j H_j - \mathrm{i}\alpha\, Re \sum_{k=0}^{N} M_{jk} P_{k+1/2} - \alpha^2 H_j = 0, \tag{25}$$

where Re is the Reynolds number as defined in KMA. The boundary conditions are:

at $\xi = -1$,

$$F(-1) = G(-1) = H(-1) = 0 \quad \text{for all } n; \tag{26}$$

at $\xi=1$,

$$\left.\begin{array}{l} F(1)=G(1)=0 \\ H'(1)=0 \end{array}\right\} \quad \text{for } n=0, \tag{27a}$$

$$\left.\begin{array}{l} F(1)\pm G(1)=0 \\ H(1)=0 \\ F'(1)=0 \text{ or } G'(1)=0 \end{array}\right\} \quad \text{for } n=\pm 1, \tag{27b}$$

$$F(1)=G(1)=H(1)=0 \quad \text{for } |n|>1. \tag{27c}$$

4. NUMERICAL SCHEME

The above equations are rearranged so that they can be represented in the generalized eigenvalue format as

$$\mathbf{DX}=\omega\mathbf{LX}, \tag{28}$$

where **D** and **L** are coefficient matrices obtained from the appropriate discretized differential operator and ω is the eigenvalue. Both **D** and **L** are square matrices with dimensions of $4N+3$. The procedure for obtaining the eigenvalues using the IMSL QZ routine is straightforward and is explained in detail by KMA.

5. RESULTS

The convergence and accuracy of the staggered Chebyshev spectral collocation method has been tested for Poiseuille flow in a pipe. In this case numerous results have already been reported by others, which greatly simplifies the comparison task.

The mean velocity for this problem is given by

$$U=0, \qquad V=0, \qquad W=1-r^2. \tag{29}$$

The linear stability of Poiseuille flow in a pipe has been studied by Metcalfe and Orszag.[13] Using a Chebyshev spectral tau method and eliminating pressure, they solved a coupled set of fourth-order and second-order equations and obtained accurate results. The convergence behaviour of the present method is compared to the Chebyshev tau formulation of Metcalfe and Orszag[13] and the non-staggered method of KMA in Table I. It is clear that Chebyshev collocation (in either form) has a much better convergence rate than the tau method. However, there is hardly any difference between the convergence rate of the two collocation formulations.

The flow in a circular pipe has been simulated spectrally by Leonard and Wray.[14] Employing shifted Jacobi polynomials, they have expanded the velocity field in a set of divergence-free functions and hence eliminated the pressure from their governing equations. They tested the convergence of their method for the linearized case of pipe flow with $n=1$, $\alpha=1$ and $Re=9600$. Table II shows a comparison of the convergence behaviour of the two collocation methods against the results of Leonard and Wray.[14] The agreement is excellent and exponential convergence is obtained. Obviously no significant round-off errors exist for any of the methods. The slight differences between the two collocation techniques and the method of Leonard and Wray[14] can be attributed to different machine precision, size of matrices, co-ordinate stretching, etc.

Table I. Convergence behaviour of the two least stable modes for Poiseuille flow in a pipe. N is the number of polynomials used to resolve each flow variable. $Re=10$, $\alpha=1$ and $n=1$

N	ω_1	ω_2
	Metcalfe and Orszag[13], Chebyshev tau formulation	
8	0·501654048 − i1·392398524	0·780474345 − i2·745100739
12	0·491070208 − i1·393534481	0·763978453 − i2·807040555
16	0·491063984 − i1·393490921	0·762026865 − i2·807291233
20	0·491064084 − i1·393490894	0·762024215 − i2·807286430
	Present calculations, staggered Chebyshev collocation	
8	0·491065536 − i1·393490713	0·762034238 − i2·807291149
12	0·491064085 − i1·393490894	0·762024225 − i2·807286425
16	0·491064084 − i1·393490894	0·762024223 − i2·807286422
20	0·491064084 − i1·393490894	0·762024223 − i2·807286422
	Non-staggered Chebyshev collocation	
8	0·491067022 − i1·393495866	0·762035973 − i2·807301022
12	0·491064084 − i1·393490896	0·762024226 − i2·807286427
16	0·491064084 − i1·393490894	0·762024223 − i2·807286422
20	0·491064084 − i1·393490894	0·762024223 − i2·807286421

Table II. Convergence behaviour of the least stable mode for Poiseuille flow in a pipe. N is the number of polynomials used to resolve each flow variable. $Re=9600$, $\alpha=1$ and $n=1$

N	ω
	Leonard and Wray[14]
22	0·95050 − i0·02313
27	0·95048142 − i0·02317074
32	0·9504813967 − i0·0231707958
37	0·9504813967 − i0·0231707958
	Present calculations, staggered Chebyshev collocation
22	0·95048 − i0·02317
27	0·95048137 − i0·02317075
32	0·9504813971 − i0·0231707938
35	0·9504813964 − i0·0231707958
37	0·9504813968 − i0·0231707957
	KMA, non-staggered Chebyshev collocation
22	0·95048 − i0·02317
27	0·95048150 − i0·02317082
32	0·9504813938 − i0·0231708010
35	0·9504813970 − i0·0231707956
37	0·9504813966 − i0·0231707958

For the same number of polynomials, the calculated eigenvalue spectrum of both collocation formulations contained identical numbers of converged eigenvalues. Increasing N results in higher numbers of converged modes, but at all times the total number of physical eigenmodes remained the same for both methods and the agreement was excellent. As N approached 48, the leading eigenvalue had converged out to 11 or 12 decimal places (which is very close to machine precision). Further increases resulted in deterioration of the accuracy starting from the leading eigenvalue. Furthermore, in the case of the staggered formulation the employment of boundary condition (4) rather than (3) resulted in no significant differences. Also, it was found that in the non-staggered case, if boundary condition (3) rather than (2) is employed, the convergence behaviour of the two collocation versions become almost identical.

Although not reported here, both collocation formulations have been compared for a variety of other flows, including the stability of a trailing line vortex, multiple-cell vortices and a rotating pipe. In each case identical results were obtained agreeing up to seven or eight significant digits. These comparisons and the above tables suggest that when Neumann pressure boundary conditions are employed correctly in hydrodynamic stability calculations (at least in the case of incompressible cylindrical flows), they do not affect the accuracy or the convergence rate of the spectral methods. It is noted further that the computational running time for the staggered code was consistently 10%–15% greater than for the non-staggered code. A simple test revealed that this increase was caused entirely by the set-up of extra interpolating matrices and not by the QZ eigenvalue solver, although for stability calculations, owing to the relatively small amount of computer time involved, this extra penalty has no practical significance.

6. CONCLUSIONS

A Chebyshev spectral collocation algorithm with staggered grid was developed to study the stability of swirling flows. The method is an extension of the technique employed by Khorrami *et al.*[1] The numerical stability problem has been formulated in primitive variable form, evaluating the velocity components at the grid points and staggering the pressure at the mid-grid points. The staggered pressure approach has eliminated the need for artificial pressure boundary conditions. Direct comparison with other formulations such as the tau method of Metcalfe and Orszag[13] and the numerical simulation of Leonard and Wray[14] has shown that the method is robust and produces accurate results. However, comparison with the non-staggered spectral method of Khorrami *et al.*[1] has shown that the use of the indirect or artificial pressure boundary conditions did not affect either the accuracy or the convergence rate of the two spectral methods. Furthermore, owing to the added complexity, the staggered code was found to be 10%–15% more expensive to run than the non-staggered version.

ACKNOWLEDGEMENTS

The author is very grateful to Dr. R. L. Ash and Dr. C. E. Grosch of Old Dominion University for helpful comments and discussions on the drafts of this paper. This work was supported by a grant from NASA Langley Research Center, NAG-1-530.

REFERENCES

1. M. R. Khorrami, M. R. Malik and R. L. Ash, 'Application of spectral collocation techniques to the stability of swirling flows', *J. Comput. Phys.*, **81**, 206–229 (1989).
2. *IMSL Library*, 9th edn, International Mathematical and Statistical Libraries Inc., 2500 City West Boulevard, Houston, TX 77042–3020, 1978.

3. S. A. Orszag and M. Israeli, 'Numerical simulation of viscous incompressible flows', *Ann. Rev. Fluid Mech.*, **6**, 281–318 (1974).
4. P. M. Gresho and R. L. Sani, 'On pressure boundary conditions for the incompressible Navier–Stokes equations', *Int. j. numer. methods fluids*, **7**, 1111–1145 (1987).
5. J. C. Strikwerda, 'Finite difference methods for the Stokes and Navier–Stokes equations', *SIAM J. Sci. Stat. Comput.*, **5**, 56–68 (1984).
6. P. J. Roache, 'A comment on the paper Finite difference methods for the Stokes and Navier–Stokes equations' by J. C. Strikwerda', *Int. j. numer. methods fluids*, **8**, 1459–1463 (1988).
7. M. R. Malik, T. A. Zang and M. Y. Hussaini, 'A spectral collocation method for the Navier–Stokes equations', *J. Comput. Phys.*, **61**, 64–88 (1985).
8. F. Montigny-Rannou and Y. Morchoisne, 'A spectral method with staggered grid for incompressible Navier–Stokes equations', *Int. j. numer. methods fluids*, **7**, 175–189 (1987).
9. M. G. Macaraeg, C. L. Streett and M. Y. Hussaini, 'A spectral collocation solution to the compressible stability eigenvalue problem', *NASA TP 2858*, 1988.
10. S. A. Orszag, 'Accurate solution of the Orr–Sommerfeld stability equation', *J. Fluid Mech.*, **50**, 689–703 (1971).
11. D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, PA, 1977.
12. D. Gottlieb, M. Y. Hussaini and S. A. Orszag, 'Theory and applications of spectral methods', in R. G. Voight, D. Gottlieb and M. Y. Hussaini (eds), *Spectral Methods for Partial Differential Equations*, SIAM, Philadelphia, PA, 1984, pp. 1–54.
13. R. W. Metcalfe and S. A. Orszag, 'Numerical calculation of the linear stability of pipe flows', *Flow Research Report No. 25*, 1973.
14. A. Leonard and A. Wray, 'New numerical method for the simulation of three-dimensional flow in a pipe', in E. Krause (ed.), *Proc. 8th Int. Conf. on Numerical Methods in Fluid Dynamics, Lecture Notes in Physics, Vol. 170*, Springer, Berlin, 1982, pp. 335–342.